The Pontryagin Construction

Riley Moriss

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Following [BM67]. The Pontryagin construction associates a (framed cobordism class of) manifold to a smooth map into a sphere. Milnor claims that this is a *generalisation of the notion of degree.*

1 Bordism

Recall that a manifold is closed if it has no boundary and is compact. Consider two closed n-dimensional submanifolds, N and N', of another closed m-dimensional manifold M.

We say that *N is cobordant to N' in M* if there is a compact manifold $X \subseteq M \times [0,1]$, such that

$$\partial X = X \cap \big(M \times \{0\} \cup M \times \{1\}\big) = N \times \{0\} \cup N' \times \{1\}$$

and moreover X extends

$$N \times [0.\epsilon) \cup N' \times (1-\epsilon, 1].$$

I wonder if I can replace this condition with something about retractions? Or if it can be removed all together and is there only for the framing condition to make sesne?

2 Framing

A *framing* of a submanifold is a smooth assignment of a basis of the normal space at each point;

$$\nu : N \to (TN^{\perp})^{m-n}.$$

A submanifold with a framing is called a *framed submanifold*.

3 Framed Bordism

Two framed submanifolds $(N, \nu), (N'\nu')$ are said to be *framed cobordent* if there is a framed cobordism (X, ν_X) between them such that the the framings agree in a neighbourhood i.e.

$$\nu_X^i(x,t) = (\nu_i(x), 0), \text{ for } (x,t) \in N \times [0, \epsilon)$$

$$\nu_X^i(x,t) = (\nu_i'(x), 0), \text{ for } (x,t) \in N \times (1-\epsilon, 1].$$

4 Pontryagin Manifold

The Pontryagin manifold is a framed manifold associated to a smooth map

$$f : M \to S^p.$$

To a smooth function we can always associate submanifolds of M given by the preimage of regular values. So pick a regular value $y \in S^p$ and we will try to put a framing on $f^{-1}(y)$.

Begin by choosing a positively oriented basis $\nu = (\nu^p, ..., \nu^p)$ of the tangent space at y TS^p_y (that is a basis "in the same direction" as the standard basis of $\mathbb{R}^n$ (the matrix transforming from one basis to the other has positive determinant)). From the inverse function theorem we have a split short exact sequence of vector spaces

$$0 \to Tf^{-1}(y)_x \to TM_x \xrightarrow{df} TS^p_y \to 0,$$

in other words

$$TM_x = TS^p_y \oplus Tf^{-1}(y)_x, \qquad \text{or} \qquad TS^p_y \cong \left(Tf^{-1}(y)_x\right)^{\perp}.$$

Pulling the chosen basis of TS^p_y back to $\left(Tf^{-1}(y)_x\right)^{\perp}$ gives the framing that we denote $\mathfrak{m} = f^*\nu$. The Pontryagin manifold of f is then

$$(f, \mathfrak{m}).$$

5 Choices

The construction made two choices, that of a regular value and that of a basis of the tangent space at that regular value. There are therefore many Pontryagin manifolds associated to a map, however there is a suitable notion of equivilence namely,

Theorem. *Given any other regular value and positively oriented basis the resulting framed manifolds are framed cobordant.*

6 Homotopy Theory

This is an interesting construction, however it is also a homotopy invariant!

Theorem. *$f, f' : M \to S^p$ are smoothly homotopic* ***if and only if*** *their Pontryagin manifolds are framed cobordant.*

7 Submanifolds

This construction also gives us a classification of compact framed submanifolds of M because

Theorem. *All compact framed submanifolds of codimension p in M are the Pontryagin manifoold of some smooth map $f : M \to S^p$.*

The proof of this theorem relies on the following important result

Lemma (Product / Tubular Neighborhood Theorem)**.** *There is a neighbourhood of N in M that is diffeomorphic to $N \times R^p$. Moreover the diffeomorphism can be chosen such that*

- *all $x \in N$ are mapped to $(x, 0) \in N \times R^p$*
- *each normal frame on N is mapped to the standard basis of R^p.*

8 Conclusion

Putting the last two results together gives us the fact that submanifolds up to framed cobordism are in bijection with smooth homotopy classes of maps!

References

[BM67] E. H. Brown and John W. Milnor. Topology from the Differentiable Viewpoint. *The American Mathematical Monthly*, 74(4):461, April 1967.